defined as , Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution

$$f(x|\theta) = 1, \quad \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$$
$$= 0, \quad \text{otherwise.}$$

a) Find the maximum likelihood estimator of θ.

b) Show that the estimator $\hat{\theta}_1 = \frac{1}{2}[X_{(1)} + X_{(n)}]$ is unbiased and has mean square error equal to

$$\frac{1}{2(n+2)(n+1)} \tag{0.1}$$

where $X_{(1)}$ is the minimum and $X_{(n)}$ is the maximum of the sample. Is $\hat{\theta}_1$ consistent?

c) Show that the statistic $\hat{\theta}_2 = X_{(n)} - \frac{1}{2}$ is consistent and find its mean square error.

d) Which of the estimators $\hat{\theta}_1$, $\hat{\theta}_2$ has lower mean square error?

[Hint: If $Z_1, \ldots, Z_n$ is a random sample from a uniform distribution on $(0, 1)$, and $V = \frac{(Z_{(1)}+Z_{(n)})}{2}$, then the density of V is given by

$$f_V(v) = n\,(2v)^{n-1} \quad \text{for } 0 < v \le \frac{1}{2}$$
$$= n\,[2\,(1-v)]^{n-1} \quad \text{for } \frac{1}{2} < v < 1$$

Moreover, $EZ_{(n)} = \frac{n}{n+1}$.]